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THE EFFECT OF ANTICLASTIC BENDING ON THE CURVATURE OF BEAMS

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Abstract—Anticlastic bending is examined and a concise presentation of the related bending deflection equations is given. Previous work is reviewed and an approximate expression is then proposed which makes it possible, with little additional labour, to include the effects of anticlastic bending in calculations of beam curvature.

NOTATION

a	characteristic of beam on elastic foundation
E	Young's modulus
F	force
h	thickness of beam
I	second moment of area
I'	second moment of area of deformed cross-section
k	elastic foundation modulus
l	length of principal beam
M	applied bending moment
R	radius of curvature of a narrow beam
t	subscript denoting the transverse beam
w	width of principal beam (i.e. length of transverse beam)
x	distance from end of transverse beam
x'	$= w-x$
Y	yield point of material
γ	width parameter
θ	angle between adjacent sections of a bent beam
κ	beam curvature including anticlastic bending effects
ν	Poisson's ratio
ρ	radius of curvature of transverse beam
$\bar{\rho}$	radius of curvature of transverse beam per unit length
ϕ	slope of the elastic line of curvature

THE BASIC EQUATIONS

A DESCRIPTION of the bending deflection of a beam is presented which includes the effects of anticlastic bending. The subject has been discussed in previous articles and this approach follows that of others, especially Searle [1]. We are concerned only with the principal beam curvature, and with obtaining a clear physical picture of how this is influenced by anticlastic curvature.

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The application of a pure moment M to a narrow prismatic bar results in a curvature described by the well-known expression:

$$\frac{1}{R} = \frac{M}{EI}. \tag{1}$$

The bending is illustrated in Fig. 1a. The curvature $1/R$ implies the existence of a traverse or anticlastic curvature of

$$\frac{1}{R_t} = \frac{-\nu}{R}. \tag{2}$$

It is observed in a wide beam, however, that the expected anticlastic curvature does not develop. For such a beam, E in equation (1) is replaced by $(E/1-\nu^2)$ in order to describe the principal deflection satisfactorily. The question then arises: what expression describes the curvature in the transitional case of a beam of "moderate" width subject to a moment M?

Assuming that anticlastic bending takes place freely, the cross-section of a beam will appear as shown in Fig. 1b. The longitudinal stresses in the beam are given by $S = My/I$ and it is clear that these stresses, normal to the plane of Fig. 1b, give rise to a net tensile

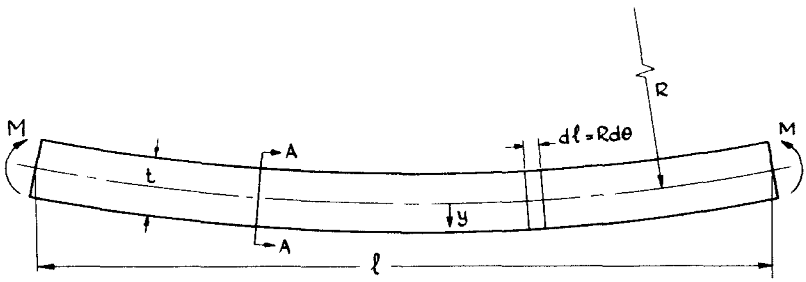

FIG. 1a. A beam bent to radius R by pure end moments M.

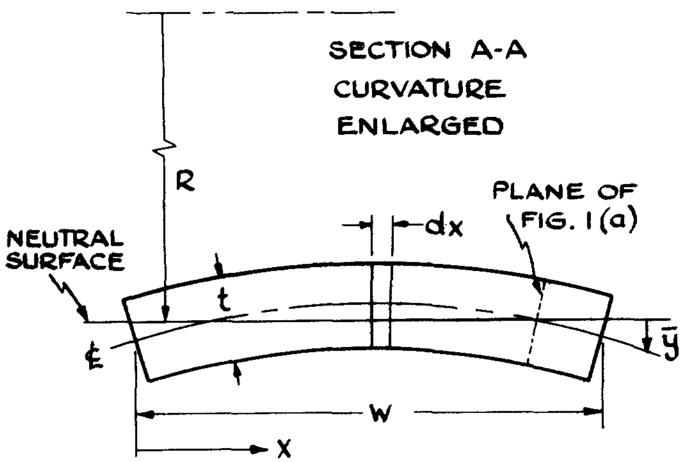

FIG. 1b. Cross-section of a bent beam showing full anticlastic curvature.

force in the cross-section of the beam near its edges and a net compressive stress in its centre. Due to the principal curvature $1/R$, adjacent cross-sections are inclined at a small angle $d\theta$ which leads to net forces towards the axis of curvature at the edges of the cross-section and away from the axis of curvature at the middle. These forces produce a moment which opposes anticlastic curvature and which must increase as the width of the beam increases.

Consider width dx of an elemental length dl of the bent beam. The radial force on this portion is

$$F = \frac{Eh\,dl\,dx}{R^2}\bar{y} \tag{3}$$

where $\bar{y}$ is the displacement of the centroid of the element from the neutral surface. Equation (3) is a typical description of an elastic foundation. Thus it follows, as others have observed, that the theory of beams on elastic foundation applies to the problem of anticlastic bending. This theory is well-known, and it is unnecessary to proceed with a detailed derivation of the appropriate equations. They are available from Hetényi [2] in a form convenient for our purposes. The elemental length dl of the principal beam may then be analysed as a "transverse" beam on elastic foundation of width dl and length w. The bending of this beam arises from the principal curvature through Poisson's ratio and, in the absence of additional externally applied moments, its end curvature is equal to $-\nu/R$. The deflection of the centre line of a finite beam on elastic foundation with defined end curvature is found from reference [2] to be:

$$\bar{y} = \frac{\nu E I_t}{R}\,\frac{2a^2}{k\,dl(\sinh aw + \sin aw)}(\sinh ax \cos ax' - \cosh ax \sin ax' + \sinh ax' \cos ax - \cosh ax' \sin ax) \tag{4}$$

where $x' = w - x$ and subscript t refers to the transverse beam. In equation (4) and subsequent expressions the origin of x is taken at the end of the beam, following Hetényi. This disregard for the symmetry of the bending is justified later by the conciseness of the resulting expressions. In equation (4), k is the foundation modulus or reaction pressure of the foundation per unit deflections, found from equation (3):

$$k = \frac{Eh}{R^2} \tag{5}$$

and a is the characteristic, a parameter involving the ratio of the foundation stiffness to the beam stiffness,

$$a = \left[\frac{k\,dl(1-\nu^2)}{4EI_t}\right]^{1/4}. \tag{6}$$

The term $(1-\nu^2)$ arises because the transverse beam is subject to lateral constraint. With the governing parameters of the elastic foundation, k and a, it is possible to express quantitatively the bending resistance of the transverse beam. From this is obtained an expression relating applied moment and curvature of the principal beam which accounts for the effects of beam width.

If no width effects occur, the application of moment M will result in curvatures given by equations (1) and (2). With the elastic foundation effect, however, the curvature $1/R_t$ will only be realized at the ends of the transverse beam. The transverse curvature elsewhere is given by $1/\rho = d^2\bar{y}/dx^2$ which from equation (4) gives

$$\frac{1}{\rho} = \frac{-\nu}{R}\frac{1}{\sinh aw + \sin aw}(\sinh ax \cos ax' + \cosh ax \sin ax' + \sinh ax' \cos ax + \cosh ax' \sin ax). \tag{7}$$

The difference between the unrestrained transverse curvature $1/R_t$ and the actual transverse curvature $1/\rho$ represents the restraining effect of the elastic foundation. Equation (7) gives a curvature which is not uniform along the length of the transverse beam. However, the average curvature is readily found from equation (7):

$$\frac{1}{\bar{\rho}} = \frac{1}{w}\int_0^w \frac{1}{p}\,dx = \frac{-\nu}{R}\frac{2}{aw}\left[\frac{\cosh aw - \cos aw}{\sinh aw + \sin aw}\right]. \tag{8}$$

Thus the restraining effect of the elastic foundation, expressed in terms of curvature, becomes

$$\frac{1}{\bar{\rho}} - \frac{1}{R_t} = \frac{\nu}{R}\left\{1 - \frac{2}{aw}\left[\frac{\cosh aw - \cos aw}{\sinh aw + \sin aw}\right]\right\}$$

using equations (2) and (8). Through Poisson's ratio this restraining curvature will affect the principal curvature by an amount

$$\frac{1}{R'} = -\nu\left\{\frac{\nu}{R}\left(1 - \frac{2}{aw}\left[\frac{\cosh aw - \cos aw}{\sinh aw + \sin aw}\right]\right)\right\}. \tag{9}$$

The final principal curvature is $\kappa = 1/R + 1/R'$ which from equations (1) and (9) gives

$$\kappa = \frac{M}{EI}(1 - \nu^2\gamma) \tag{10}$$

where γ, the anticlastic factor is defined by

$$\gamma = 1 - \frac{2}{aw}\left[\frac{\cosh aw - \cos aw}{\sinh aw + \sin aw}\right]. \tag{11}$$

The factor γ is shown in Fig. 2 as a function of aw and it is clear that for a narrow beam $\gamma \to 0$ and equation (10) gives the usual result, equation (1). For a wide beam, equation (10)

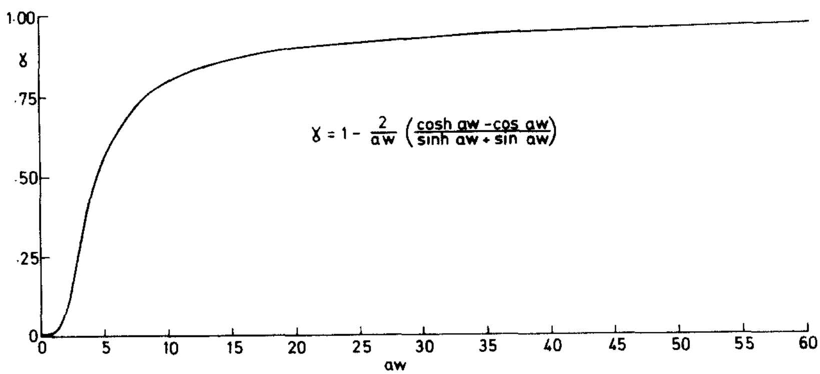

FIG. 2. The anticlastic factor γ against the characteristic times the width aw.

approaches the result for an infinitely wide beam or plate. γ changes rapidly as aw increases, but only approaches the plate value asymptotically.

REVIEW OF PREVIOUS WORK

The anticlastic curvature of bent beams was investigated by Lamb [3] and subsequently by several workers [4–8]. Ashwell [5] pointed out the applicability of beam on elastic foundation theory. Gerard [6] produced an expression equivalent to γ in the present paper and stressed the fact that narrow beams "grow" into wide ones as the radius of curvature decreases. Ashwell [5] has suggested that the second moment of area of the deformed cross section should be used when calculating principal curvature, and includes this in his expression whereas other workers ignore it. The refinement, which leads to considerable complexity in the expression for curvature, will now be examined in detail.

Referring to the deformed cross-section of Fig. 3 the second moment of area of an element of width dx and height h, with respect to the neutral surface is

$$dI' = h\left\{\frac{h^2\cos^2\phi}{12} + (\bar{y})^2\right\} dx$$

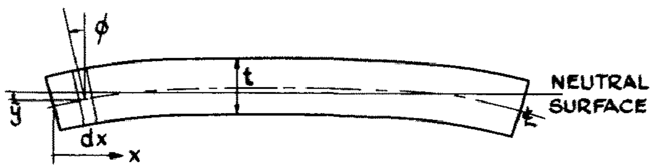

FIG. 3. Cross-section of a bent beam showing partial suppression of anticlastic curvature.

which may be expressed for the whole cross-section:

$$I' = \frac{h^3 w}{12} - \int_0^w \frac{h^3}{12}\phi^2\, dx + \int_0^w h(\bar{y})^2\, dx \tag{12}$$

where $\phi = d\bar{y}/dx$ is found from equation (4) to be

$$\phi = \frac{4EI_t a^3}{Rk\, dl}\left(\frac{\cosh ax - \cos ax}{\sinh ax + \sin ax}\right). \tag{13}$$

In other words, the second moment of area of the deformed cross-section is equal to that of the undeformed cross-section ($I_0 = h^3 w/12$) with two additional terms. Integrating equation (12) using equations (4) and (13) and evaluating in dimensionless form gives

$$\frac{I}{I_0} = 1 - I_1 + I_2$$

where

$$I_1 = \frac{1}{I_0}\int_0^w \frac{h^3}{12}\phi^2\,dx = \frac{v^2 aw\left(\frac{h}{w}\right)^2}{24(\sinh aw + \sin aw)^2}(3\sinh 2aw + 3\sin 2aw$$

$$+4aw - 4aw\cosh aw\cos aw - 6\sinh aw\sin aw - 6\cosh aw\sin aw)$$

$$I_2 = \frac{1}{I_0}\int_0^w h(\bar{y})^2\,dx = \frac{v^2}{4aw(\sinh aw + \sin aw)^2}(\sinh 2aw - \sin 2aw$$

$$-4aw\sinh aw\sin aw - 2\sinh aw\cos aw + 2\cosh aw\sin aw).$$

It remains now to assess the importance of terms I_1 and I_2 under conditions which might be encountered in practice. For a beam bent to its elastic limit, the expression relating curvature to bending stress may be written $2Y/E = h/R$. The range of values of Y/E which is of interest is from 0·001 for soft materials such as aluminium to about 0·01 for very high strength steels. The higher figure is relevant here; i.e. $h/R|_{max} = 0{\cdot}02$, from which

$$\frac{R}{w} > 50\frac{h}{w}. \tag{14}$$

From equations (5) and (6)

$$aw = \frac{1{\cdot}316(1-v^2)^{\frac{1}{4}}}{\left(\frac{R}{w}\frac{h}{w}\right)^{\frac{1}{2}}}$$

and substituting from equation (14) ($v = 0{\cdot}3$) gives

$$\frac{h}{w} > \frac{0{\cdot}183}{aw}. \tag{15}$$

Relationship (15) between aw and h/w may be used to determine the maximum value of I_1. The results of this exercise show that for all values of aw, I_1 is $< 10^{-4}$ and may be neglected compared with 1. The term I_2, however, is considerably larger than I_1. Taking the case of $v = 0{\cdot}3$ a plot of I_2 against aw is shown in Fig. 4. It is seen that I_2 has a maximum value of about 1·6 per cent of I_0 in the region of $aw = 3{\cdot}5$. The more accurate expression for bending, which replaces equation (10) and accounts for the distortion of the cross-section of the beam due to anticlastic curvature is

$$\kappa = \frac{M(1-v^2\gamma)}{\left[EI_0\left(1+\frac{v^2}{4aw(\sinh aw+\sin aw)^2}\right)(\sinh 2aw - \sin 2aw - 4aw\sinh aw\sin aw - 2\sinh aw\cos aw + 2\cosh aw\sin aw)\right]}. \tag{16}$$

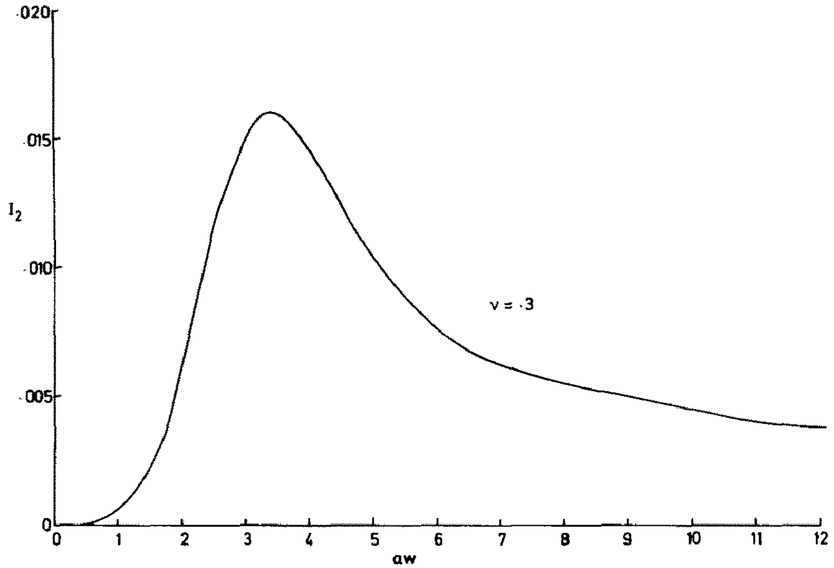

FIG. 4. Dimensionless second moment of area term I_2 against aw for Poisson's ratio = 0·3.

ALTERNATIVE EXPRESSION FOR DETERMINING CURVATURE

Equation (16) is rather complex for most structural and strength of materials applications, and the correction made in the denominator is small enough that the possibility of an approximate theory is worth examining. It will be observed that omitting the term I_2 from equation (16) will result in a curvature which is too large, whilst calculating a from equation (6) without the lateral restraint $(1-\nu^2)$ factor will result in a curvature which is too small, but the quantitative effect of these simplifications will be small. The result of using the simplified expression for curvature is shown in Fig. 5 in the form of percentage error as

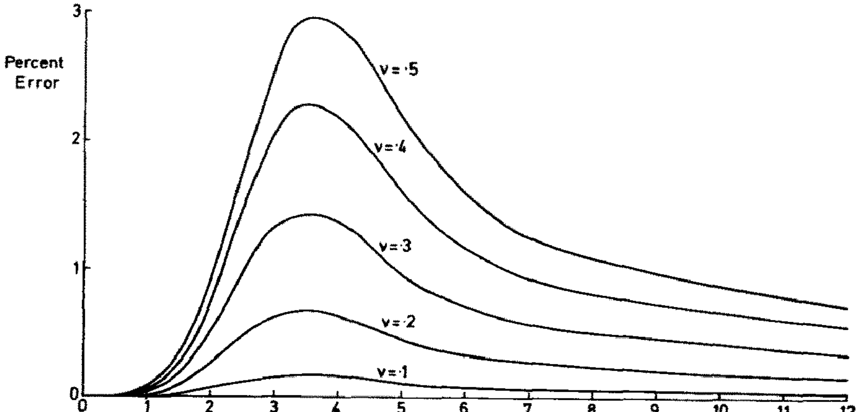

FIG. 5. Percentage error introduced by the simplified theory for beam curvature against aw at selected values of Poisson's ratio.

compared with the more accurate theory for a range of values of ν. Clearly the error introduced by the simplifications depends on the value of Poisson's ratio and reaches a maximum of about 3 per cent at $\nu = 0{\cdot}5$, $aw = 3{\cdot}5$. This represents an extreme case, however; for most materials of practical interest the value of ν will be approximately 0·3, giving an error of about 1 per cent. Such an error will be acceptable in many instances, particularly as the approximate theory underestimates the bending stiffness of the beam.

For many applications, the exact theory will be discarded on account of its complexity. Then the choice of a method for determining bending deflections is between narrow beam theory or plate theory. Reference to Fig. 2 shows that for $aw = 5$, $\gamma = 0{\cdot}5$ and thus either choice will be unavoidably in error by the same amount. Taking $\nu = 0{\cdot}3$, the unavoidable error is about 5 per cent whereas the simplified theory suggested here gives an error less than 1 per cent. It is not possible to estimate from dimensions alone whether narrow beam or plate theory is most suitable to a given situation. The possibility then exists of making a considerably larger error (twice the unavoidable error) by an inappropriate choice. The conclusion to be drawn is that by using the approximate theory presented here, curvature may be determined with as much as a tenfold increase in accuracy.

The expressions which are proposed to calculate beam curvature, then, are:

$$\kappa = \frac{M(1-\nu^2\gamma)}{EI_0}$$

where

$$\gamma = 1-\frac{2}{aw}\left(\frac{\cosh aw-\cos aw}{\sinh aw+\sin aw}\right) \tag{17}$$

$$a = 1{\cdot}316\sqrt{\left(\frac{\kappa}{h}\right)}.$$

Examination of equations (17) shows that because the parameter a is influenced by curvature a solution cannot be arrived at directly. A solution is readily obtained by iteration, however, as the convergence of aw is rapid. Therefore the use of equations (17) will not introduce much additional labour or complexity into the calculation of curvature.

APPLICATIONS OF THE SIMPLIFIED THEORY

The equations presented in this paper are based on a beam of rectangular cross-section. To apply them to other cross-sections would present difficulties if the second moment of area of the transverse beam varies with length (i.e. width of principal beam). The complexity of working with a non-prismatic transverse beam would preclude the application of this analysis unless a suitable rectangular equivalent of the cross-section could be determined.

The loading on the beam need not be confined to "pure" end moments; other forms of loading could also be considered. The beam must, however, be free to take up anticlastic curvature as outlined in the first part of the paper. If constraints, geometry or the loading distribution alter this freedom, then as Fung and Wittrich [9] point out, it will not be realistic to apply the assumptions on which equations (17) are based.

Initial beam curvature may have an important influence on the observed curvature of a loaded beam. Beams with an initial principal or transverse curvature have been discussed by Ashwell [7]. A case which is particularly suited to the use of the simplified theory arises when the initial principal curvature is large, such that the elastic curvature has a negligible effect on the value of aw, see reference [10].

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